ELSEVIER
Topology and its Applications 95 (1999) 75–83
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

"Shortest" arcs in closed planar disks vary continuously with the boundary

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Received 28 October 1996; received in revised form 3 November 1997

Abstract

Given a triple (D,a,b) where D a closed topological disk in the plane with distinct points $\{a,b\}\subset D$ there is not necessarily a path α of finite length connecting a to b in D. Nevertheless the notion of shortest arc extends naturally to the topological category as follows. There exists a *continuous* operator *geo* which assigns to each triple (D,a,b) a closed arc $\alpha\subset D$ such that α connects a and b, and moreover α has uniquely minimal finite Euclidean arclength whenever possible.

Keywords: Minimal arclength; Topological planar disk

AMS classification: Primary 58E10, Secondary 53A04

1. Introduction

Given a closed topological disk D in the plane with distinct points $\{a,b\}\subset D$, there is not necessarily a closed arc α of finite length connecting a to b in D. In this paper we establish the next best alternative: There exists a *continuous* operator *geo* which assigns to all triples (D,a,b) a closed arc $\alpha = geo(D,a,b)$ such that $\alpha\subset D$, α connects a to b, and whenever possible α has uniquely minimal finite Euclidean arclength.

We define $geo(D,a,b)=\alpha$ where α consists of those points $x\in D$ such that no chord of D separates x and $\{a,b\}$ in D. It is established by Bourgin and Renz [1] that α is indeed a closed arc in D connecting a to b and that α has minimal Euclidean arclength whenever possible.

But is *geo* itself continuous? Bourgin et al. [2] pose this question and answer it in the restricted context of isotopies with $\{a,b\}\subset \mathrm{int}(D)$.

These restrictions are in fact unnecessary and we prove that *geo* is continuous on its full domain of definition. The difficulty is that arclength of shortest paths does not vary

0166-8641/99/$ – see front matter
PII: S0166-8641(97)00275-7

continuously with the data when $\{a,b\} \cap \partial D \neq \emptyset$. Hence our approach differs from that of [2]. We establish general criteria for uniform convergence of a sequence of closed arcs (Hausdorff convergence together with uniform local connectivity) and use this criteria to verify that *geo* is indeed continuous with respect to the uniform topologies on its domain and range.

Thus for all topological disks D in the plane (including those with nowhere differentiable boundary) *geo* provides a natural interpretation of "shortest" arc between two points. This illustrates, for example, that a boat on a lake should always aim for where the opposite banks meet in order to follow the best trajectory. Continuity of *geo* ensures the existence of small homeomorphisms between ideal and actual trajectories and offsets the dangers posed by small errors in measurement. Hence, because *geo* is continuous, the *geometric* notion of geodesic is interpreted in the *topological* category of planar disks and arcs in such a way that the limit of the geodesics is the geodesic of the limit.

2. Definitions and preliminaries

If A is a compact subset of the metric space Y and X is the set of all subsets of Y homeomorphic to A we may endow X with the *uniform topology* determined by the metric

$$d(B,C) = \inf_h \Big\{ \max_{x \in B} \big\{ d\big(h(x), x\big) \mid h \text{ is a homeomorphism from } B \text{ onto } C \big\} \Big\}.$$

Alternatively recall also the *Hausdorff topology* on X determined by the metric $d(B,C) = \varepsilon$ where ε, is the smallest number such that $\forall c \in C\ \exists b \in B$ such that $d(c,b) \leqslant \varepsilon$ and $\forall b \in B\ \exists c \in C$ such that $d(c,b) \leqslant \varepsilon$. A collection of sets $A_1, A_2, \ldots \subset Y$ is said to be *uniformly locally connected* if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $\{x_n, y_n\} \subset A_n$ and $d(x_n, y_n) < \delta$ then there exists a continuum $\alpha_n \subset A_n$ such that $\{x_n, y_n\} \subset \alpha_n$ and $\operatorname{diam}(\alpha_n) < \varepsilon$.

A set $D \subset \mathbb{R}^2$ is a *closed disk* if D is homeomorphic to the closed unit disk. Any space α homeomorphic to the closed interval $[0,1]$ is said to be an *arc*. By a *chord* of the closed disk D we mean a straight line segment $C \subset D$ with endpoints $\{c,d\}$ such that $C \cap \partial D = \{c,d\}$. Recall also that if X, Y and Z are mutually disjoint subspaces of W, then Z is said to *separate X and Y in W* if X, and Y are contained in different components of $W \setminus Z$. If A is a finite subset of the metric space W then by $\operatorname{mesh}(A)$ we mean $\max_{x,y \in A} d(x,y)$.

3. The main result

We define an operator *geo* and establish its continuity in Corollary 6.

Let W, X and Y denote respectively the spaces of simple closed curves, closed disks and arcs in $\mathbb{R}^2$. We endow each of W, X and Y with the uniform topologies. Let Z denote the subspace of $X \times \mathbb{R}^2 \times \mathbb{R}^2$ consisting of all points $\{D, a, b\}$ such that $a \neq b$ and $\{a,b\} \subset D$.

Definition 1. Define $geo: Z \to Y$ as follows. Let $geo(D,a,b) = \alpha$ where α consists of those points $x \in D$ such that no chord of D separates x and $\{a,b\}$.

Remark 2. It is established [1, Thereom 4.2, p. 293] that *geo* is well defined and that $geo(D, a, b)$ has finite arclength whenever possible. Moreover if $geo(D, a, b)$ has finite arclength then it is uniquely the shortest arc connecting a and b.

Remark 3. The map $\partial : X \to W$ defined by $\partial(D) = \partial D$ is a homeomorphism. (See [3, Corollary 2.4, p. 22].) Thus convergence of a sequence of disks $D_n \to D$ is equivalent to convergence of the boundaries $\partial D_n \to \partial D$, and hence the title of our paper follows from Corollary 6.

We suppose throughout this section that $(D_n, a_n, b_n) \to (D, a, b)$ uniformly in Z.

Theorem 4. *The sequence $geo(D_n, a_n, b_n) \to geo(D, a, b)$ in the Hausdorff metric.*

Proof. Let $\beta_n = geo\{D_n, a_n, b_n\}$ and $\beta = geo\{D, a, b\}$. Suppose $\{\alpha_n\}$ is a subsequence of β_n which is convergent in the Hausdorff metric. By Lemma 7 it suffices to show $\lim_{n\to\infty} \alpha_n \subseteq \beta$. To avoid subscript notation we assume without loss of generality that $\alpha_n = \beta_n$. Choose homeomorphisms $h_n : D \to D_n$ such that $h_n \to \mathrm{id}$ uniformly. Suppose $x_n \in \alpha_n$ and $x_n \to x \in D$. Suppose, in order to obtain a contradiction, that $x \notin \beta$. By Corollary 9 choose a chord C in D which separates x from β. Let E denote the closed disk which is the closure of the component of $D \setminus C$ which contains x. Choose three planar parallel straight lines $L_1,\ L_2$, and L_3 such that L_2 is between L_1 and L_3 and such that x and C lie on different sides of L_i for each $i \in \{1, 2, 3\}$. Let $y \in C$. By Lemma 8 choose components (c_i, d_i) of $L_i \cap \mathrm{int}(E)$ such that $[c_i, d_i]$ separates x from y in E. Let $F \subset E$ be the largest closed disk such that $[c_1, d_1] \cup [c_3, d_3] \subset \partial F$. Observe that F separates x and β, and in particular F separates x and $\{a, b\}$. Let $z \in \mathrm{int}(F) \cap (c_2, d_2)$. Choose N sufficiently large such that $h_N(F)$ separates x_N and $\{a_N, b_N\}$ and $z \in \mathrm{int}(h_N(F))$. The chord C_N will separate x_N from $\{a_N, b_N\}$ where C_N denotes the chord of $h_N(F)$ contained in L_2 which separates $h_N([c_1, d_1])$ and $h_N([c_3, d_3])$. This contradicts $x_N \in \alpha_N$. $\square$

Theorem 5. *The sequence $geo\{D_n, a_n, b_n\}$ is uniformly locally connected.*

Proof. Suppose not. Let $\alpha_n = geo\{D_n, a_n, b_n\}$. Choose $\varepsilon > 0$ and pairs of points $\{x_n, y_n\} \subset \alpha_n$ such that $|x_n - y_n| \to 0$ yet such that $\mathrm{diam}(\beta_n) > \varepsilon$ where β_n denotes the subarc of α_n with endpoints x_n and y_n. Because we may pass to a subsequence if necessary we assume without loss of generality that $x_n \to x \in \alpha$. Thus

$$\left\{h_n^{-1}(x_n), h_n^{-1}(y_n)\right\} \subset D, \quad h_n^{-1}(x_n) \to x, \quad \text{and} \quad h_n^{-1}(y_n) \to x.$$

The space D is locally path connected at x. Consequently there exists a sequence of closed arcs $\gamma_n \subset D$ from $h_n^{-1}(x_n)$ to $h_n^{-1}(y_n)$ such that $\mathrm{diam}(\gamma_n) \to 0$. It follows that $h(\gamma_n) \subset D_n$ connects x_n and y_n and moreover $\mathrm{diam}(h(\gamma_n)) \to 0$. On the other hand by Lemma 11 no arc in D_n which connects x_n and y_n has shorter diameter than β_n which has diameter at least ε. This is a contradiction. $\square$

Corollary 6. *The function* $geo: Z \to Y$ *is continuous with respect to the uniform topologies on* Z *and* Y.

Proof. By Theorem 4, $geo\{D_n, a_n, b_n\} \to geo\{D, a, b\}$ in the Hausdorff metric and by Theorem 5 the sequence $geo\{D_n, a_n, b_n\}$ is uniformly locally connected. Thus by Theorem 12, $geo\{D_n, a_n, b_n\} \to geo\{D, a, b\}$ in the uniform topology. □

4. Technical lemmas

Lemma 7. *Suppose* $D \subset \mathbb{R}^2$ *is a closed disk. Suppose* $\forall n \in \mathbb{Z}^+$, α_n *is a continuum,* $\{a_n, b_n\} \subset \alpha_n \subset D$, *and suppose* α *is a closed arc in* D *with endpoints and* $\{a, b\}$. *Suppose furthermore that* $a_n \to a$ *and* $b_n \to b$. *Finally we suppose that* $\lim_{k\to\infty} \alpha_{n_k} \subseteq \alpha$ *for each convergent subsequence* (*in the Hausdorff metric*) α_{n_k}. *Then* $\lim_{n\to\infty} \alpha_n = \alpha$ *in the Hausdorff metric.*

Proof. Because D is compact, the space of closed subsets of D is compact in the Hausdorff metric. Thus it suffices to show that $\lim_{k\to\infty} \alpha_{n_k} = \alpha$ for each convergent subsequence α_{n_k} of α_n. Suppose α_{n_k} is a convergent subsequence of α_n. By hypothesis $\lim_{k\to\infty} \alpha_{n_k} \subseteq \alpha$. We observe that $\lim_{k\to\infty} \alpha_{n_k}$ is connected since connectivity is preserved when taking Hausdorff limits. Hence $\lim_{k\to\infty} \alpha_{n_k}$ is a connected subset of α which contains $\{a, b\}$. Thus $\lim_{k\to\infty} \alpha_{n_k} = \alpha$ since α itself is the only connected subset of α which contains $\{a, b\}$. □

Lemma 8. *Suppose* $D \subset \mathbb{R}^2$ *is a closed disk, suppose* $L \subset \mathbb{R}^2$ *is a straight line, suppose* x *and* y *belong to different components of* $\mathbb{R}^2 \setminus L$ *and* $\{x, y\} \subset D$. *Then there is a component* (c, d) *of* $L \cap \mathrm{int}(D)$ *such that* $[c, d]$ *separates* x *from* y *in* D.

Proof. (See Fig. 1.) We must be careful since $D \setminus L$ generally has infinitely many components and some candidates for $[c, d]$ will lead to a 'dead end'. Also, there is often more than one suitable choice for $[c, d]$. Choose an embedding $\beta: [0, 1] \hookrightarrow D$ from x to y such that $\beta(s) \in \mathrm{int}(D)$ $\forall s \in (0, 1)$. Let A denote the component of $D \setminus L$ which contains x. Let $t = \sup \beta^{-1}(A)$. Let $z = \beta(t)$. We observe that $z \neq x$ since $\beta^{-1}(A)$ has nonempty interior. We observe that $z \neq y$ since $z \in \overline{A}$ and $y \notin \overline{A}$. Thus $z \in \mathrm{int}(D)$ since $z \notin \{x, y\}$ and $\beta(s) \in \mathrm{int}(D)$ $\forall s \in (0, 1)$. Also we have $z \in L$ since $z \in \overline{A} \setminus A \subset L$. Hence $z \in L \cap \mathrm{int}(D)$. Let (c, d) denote the component of $L \cap \mathrm{int}(D)$ which contains z. The set $\partial D \cup [c, d]$ is homeomorphic to the letter θ. Let A' denote the component of $D \setminus [c, d]$ which contains x. Choose $[e, f] \subset (c, d)$ such that $\beta([0, 1]) \cap [c, d] \subset [e, f]$. Thus $[e, f] \subset \mathrm{int}(D)$. Choose $\varepsilon > 0$ such that $\bigcup_{w\in[e,f]} B(w, \varepsilon) \subset \mathrm{int}(D)$. Let

$$E = \bigcup_{w\in[e,f]} B(w, \varepsilon).$$

We observe that $E \cap A'$ is convex and in particular $E \cap A'$ is connected. Moreover $A \cap E$ is nonempty since there exists a sequence $z_n \in A$ such that $z_n \to z$. So if $w \in E \cap A'$ we may

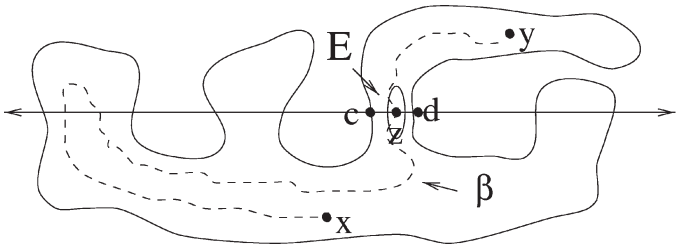

Fig. 1.

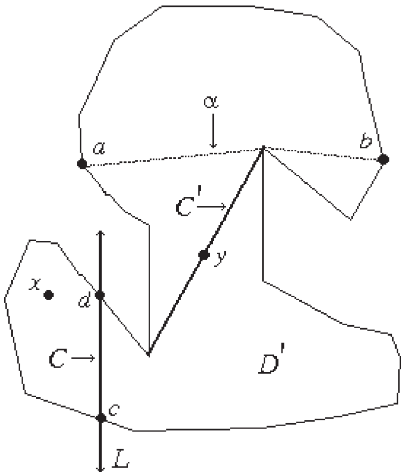

Fig. 2.

choose $z_n \in E \cap A$ such that $[w, z_n] \subset E \cap A$. It follows that $E \cap A' \subset E \cap A$. Hence if $s > t$ then $\beta(s) \notin E \cap A$ and in particular $\beta(s) \notin E \cap A'$. It follows that $\beta(s) \notin A'$ since β must pass through E upon entering A' from $[c, d]$. In particular $\beta(1) = y \notin A'$ yet $x \in A'$ and $y \notin [c, d]$. This establishes that $[c, d]$ separates x from y in D. $\square$

Corollary 9. *If $x \notin \alpha = geo\{D, a, b\}$ then there exists a chord $C \subset D$ such that C separates x and α.*

Proof. (See Fig. 2.) By definition of α choose a chord C' of D such that C' separates x and $\{a, b\}$. If we are unlucky C' will contain a point of α, and we choose another chord as follows. The set $\partial D \cup C'$ is homeomorphic to the letter θ. Let D' denote the closure of that component of $D \setminus C'$ which contains x. Thus D' is a closed disk containing both x and C'. Moreover $C' \subset \partial D' \subset C' \cup \partial D$. Choose a planar line L separating x and C' in $\mathbb{R}^2$. We observe that $L \cap \partial D' \subset L \cap \partial D$. Let y be any point on C' and apply Lemma 8 to obtain a chord $C = [c, d] \subset L$ which separates x and y in D'. Thus x and $[c, d]$ are contained in the same complementary domain of $D \setminus C'$ and hence C separates x and α in D. $\square$

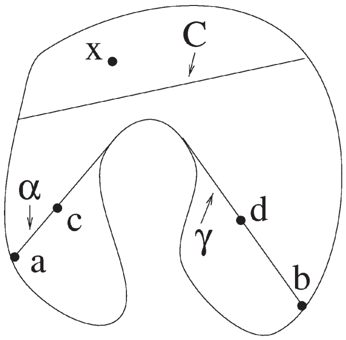

Fig. 3.

Lemma 10. *Suppose the arc γ has endpoints $\{c, d\}$ and $\gamma \subset geo(D, a, b)$. Then γ consists of those points in D which cannot be separated in D from $\{c, d\}$ by a chord.*

Proof. (See Fig. 3.) Let $\alpha = geo\{D, a, b\}$. Let β denote those points in D which cannot be separated in D from $\{c, d\}$ by a chord. By Remark 2, β is a closed arc in D with endpoints $\{c, d\}$. Suppose $x \notin \alpha$. By Corollary 9 choose a chord C which misses α and which separates x and α. Thus $x \notin \beta$ since $\{c, d\} \subset \alpha$. Hence $\beta \subset \alpha$. Thus $\beta = \gamma$ since both β and γ are subarcs of α with endpoints $\{c, d\}$. □

Lemma 11. *Suppose γ is a closed subarc of $geo(D, a, b)$ with endpoints $\{c, d\}$. Then no subcontinuum of D which contains $\{c, d\}$ has smaller diameter than γ.*

Proof. Suppose α_0 is a subcontinuum of D which contains $\{c, d\}$. We will construct a sequence $\{\alpha_n\}$ of subcontinua of D such that $\operatorname{diam}(\alpha_{n+1}) \leqslant \operatorname{diam}(\alpha_n)$ and such that $\alpha_n \to \gamma$ in the Hausdorff metric. We first observe that by Lemma 10, $\forall x \notin \gamma$ we may choose a chord $C_x \subset D$ together with a complementary domain E_x of $D \setminus C_x$ such $x \in E_x$ but $c \notin E_x$ and $d \notin E_x$. Moreover E_x is open in D, $E_x \subset D \setminus \gamma$, and the sets E_x form an open cover of $D \setminus \gamma$. Because $D \setminus \gamma$ is a separable metric space we may choose a countable subcover $E_1, E_2, \ldots$ with corresponding chords $C_1, C_2, \ldots$. Hence $\gamma = \bigcap_n^\infty (D \setminus E_n)$. We define α_{n+1} recursively as follows. Let A_n denote the convex hull of $\alpha_n \cap C_n$. Let $\alpha_{n+1} = A_n \cup (\alpha_n \setminus E_n)$. To see that α_{n+1} is connected let $\alpha_n \setminus E_n = \bigcup \alpha_{n_i}$ where $\forall i$, α_{n_i} is a component $\alpha_n \setminus E_n$. If $A_n = \emptyset$ then $\alpha_{n+1} = \alpha_n$ which is connected. If $A_n \neq \emptyset$ then $\forall i$, $\alpha_{n_i} \cap A_n \neq \emptyset$, $\alpha_{n+1} = A_n \cup_i \alpha_{n_i}$, and hence α_{n+1} is connected since its the union of continua each of which intersects the continuum A_n. Moreover $\operatorname{diam}(\alpha_{n+1}) \leqslant \operatorname{diam}(\alpha_n)$ and $\alpha_{n+1} \cap (\bigcup_{i=1}^n E_i) = \emptyset$. The latter property ensures that each subsequential limit of α_n is contained in γ. Hence by Lemma 7, $\alpha_n \to \gamma$. It follows that $\operatorname{diam}(\alpha_0) \geqslant \operatorname{diam}(\gamma)$. Consequently no subcontinuum of D which contains γ has smaller diameter than that of γ. □

5. Criteria for uniform convergence of arcs

In this section we establish that Hausdorff convergence together with uniform local connectivity is sufficient to guarantee uniform convergence of a sequence of arcs $\alpha_n \to \alpha$ in the metric space W. The converse holds more generally when α_n and α are homeomorphic locally path connected compacta. When $W = \mathbb{R}^2$ (as in the case of *geo*) the convergence criteria can also be deduced from results on conformal maps and prime ends (see Pommerenke [3, pp. 12, 14, 22]), although this invokes far more machinery than necessary.

Throughout this section we suppose that W is a metric space and that each of α_n and α is a closed arc in W.

Theorem 12. *The sequence of arcs $\alpha_n \to \alpha$ in the uniform topology if and only if $\alpha_n \to \alpha$ in the Hausdorff metric and $\{\alpha_n\}$ is uniformly locally connected.*

Proof. One direction holds easily in much more general contexts and is the content of Remark 13. The other direction is established in Lemma 17. □

Remark 13. Suppose A is a locally connected compactum in the metric space W, $A_n \subset W$, and $A_n \to A$ in the uniform topology. Then $\{A_n\}$ is uniformly locally connected and $A_n \to A$ in the Hausdorff topology.

Lemma 14. *Suppose $\alpha_n \to \alpha$ in the Hausdorff metric and a, c, b occur consecutively on α. Suppose $\{a_n, b_n\} \subset \alpha_n$ and $a_n \to a$ and $b_n \to b$. Let γ_n denote the subarc of α_n with endpoints $\{a_n, b_n\}$. Then there exists a sequence $c_n \in \gamma_n$ such that $c_n \to c$.*

Proof. (See Fig. 4.) Because α is an absolute retract (Tietze extension theorem) let $r : W \to \alpha$ be a retraction such that $r(a_n) = a$ and $r(b_n) = b$. Let $V = \alpha \cup \alpha_1 \cup \alpha_2 \cdots$. Suppose $\varepsilon > 0$. It suffices to prove that there exists N such that if $n \geqslant N$ then $B(c, \varepsilon) \cap \gamma_n \neq \emptyset$. To make things harder on ourselves we may assume that ε is sufficiently small that a and b belong to different components of $\alpha \setminus B(c, \varepsilon/2)$. Because V is compact $r|_V$ is uniformly continuous. Hence we may choose $\delta < \varepsilon/4$ such that if $\{x, y\} \subset V$ and $d(x, y) < \delta$ then $d(r(x), r(y)) < \varepsilon/4$. Choose N so that if $n \geqslant N$ then $H(\alpha_n, \alpha) < \delta$. Suppose $x \notin B(c, \varepsilon)$, $n \geqslant N$ and $x \in \gamma_n$. Choose $y \in \alpha$ such that $d(x, y) < \delta$. Hence

$$d\big(x, r(x)\big) \leqslant d(x, y) + d\big(y, r(y)\big) + d\big(r(y), r(x)\big) < \delta + 0 + \varepsilon/4 < \varepsilon/2. \tag{1}$$

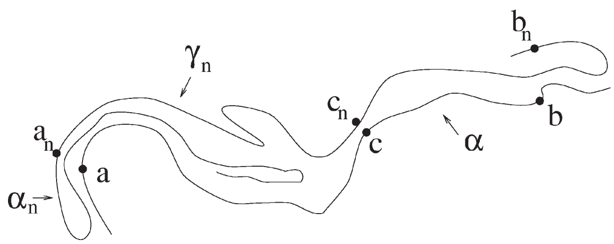

Fig. 4.

Consequently

$$d\big(c, r(x)\big) \geqslant d(c, x) - d\big(x, (r(x)\big) > \varepsilon - \varepsilon/2 = \varepsilon/2. \tag{2}$$

Thus we would have a contradiction if $B(c,\varepsilon) \cap \gamma_n = \emptyset$ since r would then map the connected space γ_n into at least two components of the disconnected space $\alpha \setminus B(c, \varepsilon/2)$. $\square$

Lemma 15. *Suppose $\alpha_n \to \alpha$ in the Hausdorff metric and α_n is uniformly locally connected. Then if $x_1,\ x_2,\ x_3$ are consecutive points on α there exists $\delta > 0$ and $N \in \mathbb{Z}^+$ such that $\forall n \geqslant N$ if $\{y_{n_1}, y_{n_2}, y_{n_3}\} \subset \alpha_n$ and $d(y_{n_i}, x_i) < \delta$ then $y_{n_1},\ y_{n_2},\ y_{n_3}$ occur consecutively on α_n.*

Proof. Suppose $x_1,\ x_2,\ x_3$ are consecutive points of α. In order to obtain a contradiction we suppose that the conclusion of the lemma does not hold. Hence we may construct a subsequence$\{\beta_n\}$ of $\{\alpha_n\}$ together with $\{y_{n_1}, y_{n_2}, y_{n_3}\} \subset \beta_n$ such that $\forall i,\ \ y_{n_i} \to x_i$, yet y_{n_2} is not between y_{n_1} and y_{n_3} on β_n. Thus on β_n, $\forall n$ either y_{n_1} is between y_{n_2} and y_{n_3} or y_{n_3} is between y_{n_1} and y_{n_2}. To avoid passing to yet another subsequence we assume without loss of generality that $\forall n,\ \ y_{n_1}$ is between y_{n_2} and y_{n_3} on β_n. Thus by Lemma 14 there exists $z_n \in [y_{n_1}, y_{n_3}]_{\beta_n}$ such that $z_n \to x_2$. It follows that $d(z_n, y_{n_2}) \to 0$ and $\mathrm{diam}[z_n, y_{n_2}] \to 0$. On the other hand $d(y_{n_1}, y_{n_2}) \leqslant \mathrm{diam}[z_n, y_{n_2}]$ since y_{n_1} is between z_n and y_{n_2} on β_n. Therefore $\mathrm{diam}[z_n, y_{n_2}]$ is eventually bounded below by $d(x_1, x_2)$ since $d(y_{n_1}, y_{n_2}) \to d(x_1, x_2)$. This contradicts the fact that $\mathrm{diam}[z_n, y_{n_2}] \to 0$. $\square$

Corollary 16. *Suppose $\alpha_n \to \alpha$ in the Hausdorff metric and α_n is uniformly locally connected. Then if $x_1, x_2, x_3, \ldots, x_m$ are consecutive points on α there exists $\delta > 0$ and $N \in \mathbb{Z}^+$ such that $\forall n \geqslant N$ if $\{y_{n_1}, y_{n_2}, y_{n_3}, \ldots, y_{n_m}\} \subset \alpha_n$ and $d(y_{n_i}, x_i) < \delta$ then $y_{n_1}, y_{n_2}, y_{n_3}, \ldots, y_{n_m}$ occur consecutively on α_n.*

Proof. Apply the previous to lemma to each of $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots,$ and $\{x_{m-2}, x_{m-1}, x_{m-2}\}$. $\square$

Lemma 17. *Suppose $\alpha_n \to \alpha$ in the Hausdorff metric and α_n is uniformly locally connected. Then $\alpha_n \to \alpha$ in the uniform topology.*

Proof. Choose a sequence of partitions $\{x_{n_1}, x_{n_2}, \ldots, x_{n_n}\} \subset \alpha$ such that $\{x_{n_1}, x_{n_2}, \ldots, x_{n_n}\}$ converges to α in the Hausdorff metric and such that x_{n_i} is not an endpoint of α. Let $N_0 = 1$. Beginning with $n = 1$ we define $N_n,\ \delta_n$ and δ_n' recursively as follows. Applying the hypotheses of uniform local connectivity choose $\delta_n' < 1/n$ such that if $\{y, z\} \subset \alpha_m$ and $d(y, z) < \delta_n'$ then $\mathrm{diam}([y, z]_{\alpha_m}) < 1/n$. By Corollary 16 choose $\delta_n < \delta_n'$ and $M_n > N_{n-1}$ so that if $m \geqslant M_n$ and $\{y_{m_1}, y_{m_2}, \ldots, y_{m_n}\} \subset \alpha_m$ and $\forall i,\ d(x_{n_i}, y_{m_i}) < \delta_n$ then $y_{m_1}, y_{m_2}, \ldots, y_{m_n}$ occur consecutively on α_m. Choose $N_n \geqslant M_n$ so that $H(\alpha_m, \alpha) < \delta_n$ when $m \geqslant N_n$. Suppose $N_n \leqslant m < N_{n+1}$. Choose $\{y_{m_1}, y_{m_2}, \ldots, y_{m_n}\} \subset \alpha_m$ such that $d(x_{n_i}, y_{m_i}) < \delta_n$ and such that y_{m_i} is not an endpoint of α_m. Define $\beta_m(x_{n_i}) = y_{m_i}$. Because $\{y_{m_1}, y_{m_2}, \ldots, y_{m_n}\}$ occur consecutively and are not endpoints we extend β_m

to a homeomorphism $\beta_m : \alpha \to \alpha_m$. It follows moreover that $\beta_m \to \mathrm{id}_\alpha$ uniformly since $\mathrm{diam}[x_{n_i}, x_{n_{i+1}}]_\alpha \to 0$ and $\mathrm{diam}[y_{m_i}, y_{m_{i+1}}] \to 0$ as $n, m \to \infty$. $\square$

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